Some error bounds for Gauss–Jacobi quadrature rules

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Abstract

We estimate the error of Gauss–Jacobi quadrature rule applied to a function f, which is supposed locally absolutely continuous in some Besov type spaces, or of bounded variation on $[-1, 1]$. In the first case the error bound concerns the weighted main part φ–modulus of smoothness of f introduced by Z. Ditzian and V. Totik, while in the second case we deal with a Stieltjes integral with respect to f. The stated estimates generalize several error bounds from literature and, in both the cases, they assure the same convergence rate of the error of best polynomial approximation in weighted L^1 space.

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1. Introduction

Gauss–Jacobi quadrature rules have been extensively studied in literature (see e.g. [1, 4, 10, 14] and the references therein). For a given $n \in \mathbb{N}$, they provide the following approximation

$$\int_{-1}^{1} f(x)u(x)dx \approx \sum_{k=1}^{n} \lambda_k f(x_k),$$

where $u(x) = v^{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, is a given Jacobi weight, $p_n(u, x)$ is the associated orthonormal Jacobi polynomial of degree n and positive leading coefficient,

$$\lambda_k := \left[\sum_{j=0}^{n-1} p_j(u, x_k)^2\right]^{-1}, \qquad k = 1, \dots, n,$$

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are the well–known Christoffel numbers, and $x_1 < x_2 < \ldots < x_n$ are the zeros of $p_n(u,x)$. We assume that $f \in L^1_u := \{f : \ \|fu\|_1 := \int_{-1}^1 |f(x)|u(x)dx < \infty\}$ is defined at these nodes and locally bounded in $[-1,1]$ (i.e. bounded in each $[a,b] \subseteq (-1,1)$).

Set

$$R_n(f)_u := \left| \int_{-1}^1 f(x)u(x)dx - \sum_{k=1}^n \lambda_k f(x_k) \right|,$$

and denoted by $\mathbb{P}_n$ the set of all polynomials of degree at most n, it is well known that

$$R_n(f)_u = 0, \qquad \forall f \in \mathbb{P}_{2n-1}, \tag{1}$$

while, in the general case, $R_n(f)_u \to 0$ as $n \to \infty$ and the rate of convergence depends on the smoothness properties of f. Regarding this, various error estimates have been proved by several authors under different smoothness assumptions of the integrand function (see e.g. [5, 6, 9, 12, 15, 16, 17, 18, 25, 31]).

In particular, for the non–weighted case (i.e. $u(x) = 1$), De Vore and Scott [5] proved that if for some integer $s \le 2n$ we have $\|f^{(s)}\varphi^s\|_1 < \infty$, being here and in the following $\varphi(x) := \sqrt{1-x^2}$, then we have

$$R_n(f) \le \frac{C_s}{n^s} \|f^{(s)}\varphi^s\|_1, \tag{2}$$

where $R_n(f)$ denotes the quadrature error when $u = 1$ and C_s is a positive constant depending on s, but independent of f and n.

Later on, Ditzian and Totik [7, Section 7.4] combined their results with (2) and stated an error bound based on special moduli of smoothness rather than on the derivatives of f. More precisely, they proved that

$$R_n(f) \le \frac{M_r}{n} \int_0^{\frac{1}{n}} \frac{\omega^r_\varphi(f,t)}{t^2} dt, \qquad n > r, \tag{3}$$

holds for all $f \in L^1$, where $M_r > 0$ is independent of n and f, and

$$\omega^r_\varphi(f,t) := \sup_{0<h\le t} \|\Delta^r_{h\varphi} f\|_1,$$

being $\Delta^r_{h\varphi} f$ the central rth difference of f of variable step size $h\varphi(x)$.

Inspired by Ditzian–Totik results, in Section 2 (Theorem 1) we state the following estimate

$$R_n(f)_u \le \frac{\mathcal{C}}{n} \int_0^{\frac{1}{n}} \frac{\Omega^r_\varphi(f,t)_u}{t^2} dt, \qquad n > r, \qquad \mathcal{C} \ne \mathcal{C}(n,f), \tag{4}$$

where [7, eq.(8.1.2)]

$$\Omega^r_\varphi(f,t)_u := \sup_{0<h\le t} \|(\Delta^r_{h\varphi} f)u\|_{L^1[-1+2r^2h^2, 1-2r^2h^2]},$$

throughout the paper $\mathcal{C}$ denotes a positive constant which can take different values in different formulas, and $\mathcal{C} \neq \mathcal{C}(n, f)$ means that $\mathcal{C}$ does not depend on n and f.

Comparing with (3), we observe that (4) concerns the more general weighted case and, instead of the complete modulus ω_φ^r, it regards the so–called main part modulus Ω_φ^r. We recall that, in the weighted case, $\omega_\varphi^r(f,t)_u$ is defined only for Jacobi weights $u = v^{\alpha,\beta}$ with $\alpha, \beta \geq 0$ (see [7, Remark 6.1.2]), while in (4) $\Omega_\varphi^r(f,t)_u$ results defined without any restriction on $u \in L^1$ and it is often easier to compute, since it does not take into account of the function values close to the extremes ± 1 (see [7, Section 8.5] for some examples and calculations).

Moreover, the complete and main part moduli are related by [7, Theorem 6.2.2]

$$\mathcal{C}^{-1}\Omega_\varphi^r(f,t)_u \leq \omega_\varphi^r(f,t)_u \leq \mathcal{C}\int_0^t \frac{\Omega_\varphi^r(f,\tau)_u}{\tau} d\tau, \qquad \mathcal{C} \neq \mathcal{C}(f,t),$$

and they both well characterize the rate of convergence to zero of the error of best polynomial approximation

$$E_n(f)_u := \inf_{P\in\mathbb{P}_n} \|(f-P)u\|_1, \tag{5}$$

in terms of the smoothness of f. In particular, we have [7, Corollary 8.2.2]

$$E_n(f)_u = \mathcal{O}(n^{-a}) \Longleftrightarrow \Omega_\varphi^r(f,t)_u = \mathcal{O}(t^a), \qquad r > a > 0. \tag{6}$$

Hence, from (4) we deduce that if $E_n(f)_u = \mathcal{O}(n^{-a})$ holds for any $a > 1$, then $R_n(f)_u = \mathcal{O}(n^{-a})$ holds too.

Indeed this result was already known in Sobolev spaces

$$W_u^s = \{f \in L_u^1 : \|f^{(s)}\varphi^s u\|_1 < \infty\}, \qquad s \in \mathbb{N}, \tag{7}$$

where $E_n(f)_u = \mathcal{O}(n^{-s})$ holds [23] and $R_n(f)_u = \mathcal{O}(n^{-s})$ follows from the weighted version of (2) (namely (16), which we deduced from (4), but it was already proved [19, Th. 5.1.8]).

The novelty introduced by (4) mainly concerns functions in the intermediate Besov spaces defined by

$$B_u^s := \left\{ f \in L_u^1 : \int_0^1 \frac{\Omega_\varphi^r(f,t)_u}{t^{s+1}} dt < \infty \right\}, \qquad r > s \geq 1. \tag{8}$$

These spaces were introduced in [8] for all $s > 0$, but we take $s \geq 1$ since this assures the convergence of the integral on the right–hand side in (4). We recall that [8] Besov spaces have the peculiarity to be characterized by the best polynomial approximation in L_u^1. Similarly to Sobolev spaces, for all $f \in B_u^s$ we have [7] $E_n(f)_u = \mathcal{O}(n^{-s})$, but Besov spaces are defined also for non–integer values of $s > 1$ and they satisfy $W_u^{[s]+1} \subseteq B_u^s \subseteq W_u^{[s]}$, where, as usual, $[s]$ denotes the integer part of s.

Then for all functions $f \in B^s_u$ with $s > 1$, which are not sufficiently smooth to belong to $W_u^{[s]+1}$, we remark that the new bound (4) yields $R_n(f)_u = \mathcal{O}(n^{-s}) = E_n(f)_u$, while using the already known estimates in Sobolev spaces, we are able only to get $R_n(f)_n = \mathcal{O}(n^{-[s]})$.

Nevertheless, when $\Omega^r_\varphi(f,t)_u \sim t^a$ holds for some $0 < a \le 1$, we have that $f \in B^s_u$ for any $0 < s < a$ but $f \notin B^1_u$ and both the estimates (3) and (4) are not significant because the integrals on their right–hand side do not converge. Indeed, the convergence of these integrals implies that f is L^1 equivalent to a locally absolutely continuous function [7, Theorem 6.3.1], and we are currently working on getting an estimate of the following type

$$R_n(f)_u \le \mathcal{C} \int_0^{\frac{1}{n}} \frac{\omega^r_\varphi(f,t)_u}{t} dt, \qquad n > r, \qquad \mathcal{C} \neq \mathcal{C}(n,f), \tag{9}$$

which should assure, even when $0 < a \le 1$, that $E_n(f)_u = \mathcal{O}(n^{-a})$ implies $R_n(f)_u = \mathcal{O}(n^{-a})$.

For the time being, (9) remains an open problem and a partial result is given in Section 3, where we examine the case that f is of bounded variation on $[-1,1]$. In this case $\Omega^r_\varphi(f,t)_u \sim t$ holds and we state that $R_n(f)_u = \mathcal{O}(n^{-1})$ by means of an estimate based on the Stieltjes integral w.r.t. f (cf. (17)). This result extends to the weighted case a previous bound stated in [12]. It has been proved by using certain delayed means of the Fourier projections (de la Vallée Poussin means), which approximation properties will be briefly recalled.

2. The case of locally absolutely continuous functions

Let f be a locally absolutely continuous function in $[-1,1]$ (i.e. absolutely continuous in each $[a,b] \subseteq (-1,1)$). We suppose f can be also unbounded at the extremes ± 1, but such that $\|fu\|_1 < \infty$. We aim to state a general error bound for $R_n(f)_u$, which yields $R_n(f)_u = \mathcal{O}(n^{-a})$ under the assumption that $E_n(f)_u = \mathcal{O}(n^{-a})$ holds for some $a > 1$, being $E_n(f)_u$ defined by (5).

To this aim we use Ditzian–Totik results in [7], which allow us to measure the rate of convergence of $E_n(f)_u$ by the main part φ–modulus of smoothness of f, defined as follows [7, eq. (8.1.2)]

$$\Omega^r_\varphi(f,t)_u := \sup_{0<h\le t} \|(\Delta^r_{h\varphi} f)u\|_{L^1(I_{r,h})}, \qquad I_{r,h} := [-1+2r^2h^2, 1-2r^2h^2],$$

where $\|g\|_{L^1[a,b]} := \int_a^b |g(x)|dx$, and $\Delta^r_{h\varphi} f$ denotes the central rth difference of f of variable step size $h\varphi(x) = h\sqrt{1-x^2}$, i.e.

$$\Delta^r_{h\varphi} f(x) := \sum_{k=0}^r (-1)^k \begin{pmatrix} r \\ k \end{pmatrix} f\left(x + \left(\frac{r}{2} - k\right) h\varphi(x)\right).$$

In [7, eq. (8.2.4)] it is stated that there exists a sequence of approximating polynomials for f, namely $P^*_n \in \mathbb{P}_n$, such that

$$\|(f - P^*_n)u\|_{L^1[-1+cn^{-2}, 1-cn^{-2}]} \le \mathcal{C}\ \Omega^r_\varphi\left(f, \frac{1}{n}\right)_u, \qquad r < n, \tag{10}$$

holds, where the constant $\mathcal{C} > 0$ depends on $c > 0$, but $\mathcal{C} \neq \mathcal{C}(n, f)$.

Moreover (cf. [7, pp. 94-95]) on the whole interval $[-1, 1]$ the previous polynomials P_n^* satisfy

$$\|(f - P_n^*)u\|_1 \leq \mathcal{C} \int_0^{\frac{1}{n}} \frac{\Omega_\varphi^r(f,t)_u}{t}\, dt, \qquad \mathcal{C} \neq \mathcal{C}(n, f), \tag{11}$$

and we also have

$$f(x_k) - P_n^*(x_k) = \sum_{j=0}^{\infty} \left[P_{2^{j+1}n}^*(x_k) - P_{2^j n}^*(x_k)\right], \qquad k = 1, \dots, n. \tag{12}$$

We remark that the local absolute continuity of f assures that both the right–hand sides in (11) and (12) converge and that (12) is satisfied at any system of Jacobi zeros x_k (in the general case it holds a.e. in $[-1, 1]$).

By means of the previous results, we state the following

Theorem 1. *For all Jacobi weights $u \in L^1$, each locally absolutely continuous function $f \in L_u^1$ and any pair of integers $n > r$, we have*

$$R_n(f)_u \leq \frac{\mathcal{C}}{n} \int_0^{\frac{1}{n}} \frac{\Omega_\varphi^r(f,t)_u}{t^2} dt, \qquad \mathcal{C} \neq \mathcal{C}(n, f). \tag{13}$$

Proof of Theorem 1. We consider the non–trivial case that the integral on the right hand side in (13) converges. Then, by the previous discussion, there exists a polynomial $P_n^* \in \mathbb{P}_n$ satisfying (10)–(12). Hence, by (1) and (11), we get

$$\begin{aligned}
R_n(f)_u &= R_n(f - P_n^*)_u \leq \|(f - P_n^*)u\|_1 + \sum_{k=1}^{n} \lambda_k |(f - P_n^*)(x_k)| \\
&\leq \mathcal{C} \int_0^{\frac{1}{n}} \frac{\Omega_\varphi^r(f,t)_u}{t} dt + \sum_{k=1}^{n} \lambda_k \left| \sum_{j=0}^{\infty} (P_{2^{j+1}n}^* - P_{2^j n}^*)(x_k) \right| \\
&\leq \frac{\mathcal{C}}{n} \int_0^{\frac{1}{n}} \frac{\Omega_\varphi^r(f,t)_u}{t^2} dt + \sum_{j=0}^{\infty} \sum_{k=1}^{n} \lambda_k |(P_{2^{j+1}n}^* - P_{2^j n}^*)(x_k)|,
\end{aligned}$$

and in order to complete the proof, we are going to prove that

$$\sum_{j=0}^{\infty} \sum_{k=1}^{n} \lambda_k |(P_{2^{j+1}n}^* - P_{2^j n}^*)(x_k)| \leq \frac{\mathcal{C}}{n} \int_0^{\frac{1}{n}} \frac{\Omega_\varphi^r(f,t)_u}{t^2}\, dt.$$

To this aim we apply the following Marcinkiewicz type inequality (see e.g. [20, Theorem 2.6])

$$\sum_{k=1}^{n} \lambda_k |(P_{ln}(x_k)| \leq \mathcal{C} l \|P_{ln} u\|_1, \qquad \forall P_{ln} \in \mathbb{P}_{ln}, \qquad \mathcal{C} \neq \mathcal{C}(l, n, P), \tag{14}$$

and Remez type inequality (see e.g. [7, p. 91, (B)])

$$\|P_n u\|_1 \leq \mathcal{C}\|P_n u\|_{L^1\left[-1+\frac{\mathcal{C}}{n^2},1-\frac{\mathcal{C}}{n^2}\right]}, \qquad \forall P_n \in \mathbb{P}_n, \quad \mathcal{C} \neq \mathcal{C}(n,P), \tag{15}$$

in order to get

$$\begin{aligned} &\sum_{j=0}^{\infty}\sum_{k=1}^{n} \lambda_k |(P^*_{2^{j+1}n} - P^*_{2^j n})(x_k)| \leq \mathcal{C}\sum_{j=0}^{\infty} 2^j \|(P^*_{2^{j+1}n} - P^*_{2^j n})u\|_1 \\ \leq\ & \mathcal{C}\sum_{j=0}^{\infty} 2^j \|(P^*_{2^{j+1}n} - P^*_{2^j n})u\|_{L^1[-1+\mathcal{C}(2^{j+1}n)^{-2},1-\mathcal{C}(2^{j+1}n)^{-2}]}. \end{aligned}$$

Then, by using (10) and the following properties [7]

$$\begin{aligned} &(i) \quad \Omega^r_\varphi(f,2t)_u \leq \mathcal{C}2^r\Omega^r_\varphi(f,t)_u, \\ &(ii) \quad t_1 \leq t_2 \implies \Omega^r_\varphi(f,t_1)_u \leq \Omega^r_\varphi(f,t_2)_u, \end{aligned}$$

we conclude the previous estimate as follows

$$\begin{aligned} &\sum_{j=0}^{\infty}\sum_{k=1}^{n} \lambda_k |(P^*_{2^{j+1}n} - P^*_{2^j n})(x_k)| \\ \leq\ & \mathcal{C}\sum_{j=0}^{\infty} 2^j \left[\Omega^r_\varphi\left(f, \frac{1}{2^{j+1}n}\right)_{u,1} + \Omega^r_\varphi\left(f, \frac{1}{2^j n}\right)_u\right] \\ \leq\ & \mathcal{C}\sum_{j=0}^{\infty} 2^j\ \Omega^r_\varphi\left(f, \frac{1}{2^{j+1}n}\right)_u \\ \leq\ & \frac{\mathcal{C}}{n}\sum_{j=0}^{\infty} (2^j n)^2\ \Omega^r_\varphi\left(f, \frac{1}{2^{j+1}n}\right)_u \left[\int_{\frac{1}{2^{j+1}n}}^{\frac{1}{2^j n}} dt\right] \\ \leq\ & \frac{\mathcal{C}}{n}\sum_{j=0}^{\infty} \int_{\frac{1}{2^{j+1}n}}^{\frac{1}{2^j n}} \frac{\Omega^r_\varphi(f,t)_u}{t^2}\ dt = \frac{\mathcal{C}}{n}\int_0^{\frac{1}{n}} \frac{\Omega^r_\varphi(f,t)_u}{t^2}\ dt. \end{aligned}$$

□

We observe that, if for some integer $s > 1$ the function f is such that $f^{(s-1)}$ is locally absolutely continuous on $[-1, 1]$ and $\|f^{(s)}\varphi^s u\|_1 < \infty$, namely $f \in W^s_u$ (cf. (7)), then we have [7, Lemma 8.1.2]

$$\Omega^s_\varphi(f,t)_u \leq \mathcal{C}t^s\|f^{(s)}\varphi^s u\|_1, \qquad \mathcal{C} \neq \mathcal{C}(n,f),$$

and Theorem 1 implies the already known estimate

$$R_n(f)_u \leq \frac{\mathcal{C}}{n^s}\|f^{(s)}\varphi^s u\|_1, \qquad \mathcal{C} \neq \mathcal{C}(n,f), \qquad \forall f \in W^s_u, \tag{16}$$

which has been proved in [19, Th. 5.1.8] for $s = 1$ too. Nevertheless, for some specific functions in Sobolev spaces, the new estimate in Theorem 1 improves the existing bounds, as shown in the next example.

Example 1. *Take* $f(x) = \log(1-x^2)$ *and* $u(x) = (1-x^2)^{-\frac{1}{3}}$. *In this case* $f \in W_u^1 - W_u^2$ *and we have* $\Omega_\varphi^r(f,t)_u \sim t^{\frac{4}{3}}$ *(see [7, p. 110]). Hence (13) implies that*

$$R_n(f)_u = \mathcal{O}(n^{-\frac{4}{3}}).$$

Note that this result cannot be deduced from the known estimates in Sobolev spaces (where (16) yields only $R_n(f)_u = \mathcal{O}(n^{-1})$*), neither other existing error bounds with the weighted uniform norm are applicable, being* $\|fu\|_\infty = \infty$.

More generally, we can say that Theorem 1 provides significative results in Besov type subspaces of L_u^1 defined by (8). For all $s > 1$, the functions $f \in B_u^s$ are so smooth to satisfy $\Omega_\varphi^r(f,t)_u = \mathcal{O}(t^s) = E_n(f)_u$, hence the integral on the right–hand side of (13) converges, and Theorem 1 yields $R_n(f)_u = \mathcal{O}(n^{-s})$.

In conclusion, the next corollary follows from Theorem 1 by virtue of (6).

Corollary 1. *If* $f \in L_u^1$ *is such that* $E_n(f)_u = \mathcal{O}(n^{-s})$ *holds for some* $s > 1$, *then we have* $R_n(f)_u = \mathcal{O}(n^{-s})$.

3. The case of functions of bounded variation

Throughout this section we suppose that $f \in L_u^1$ is a function of bounded variation and, as usual, we assume that

$$f(x_k) = \frac{f(x_k)^+ + f(x_k)^-}{2}, \qquad f(x_k)^\pm := \lim_{x \to x_k^\pm} f(x).$$

For such a function it is well–known that $\Omega_\varphi^r(f,t)_u \sim t$ holds , hence (13) is trivial being the right–hand side equal to infinity. By the next theorem, we state a different estimate, which assures, for all functions f of bounded variation, that the Gaussian quadrature error tends to zero with the same convergence rate of the error of best polynomial approximation in L_u^1, namely $R_n(f)_u = \mathcal{O}(n^{-1})$. In the special case $u(x) = 1$ a similar result has been proved in [12].

Theorem 2. *For all functions* f *of bounded variation on* $[-1,1]$ *and any* $u \in L^1$, *we have*

$$R_n(f)_u \le \frac{\mathcal{C}}{n} \int_{-1}^{1} u(t)\varphi(t)|df(t)|, \qquad \mathcal{C} \neq \mathcal{C}(n,f). \tag{17}$$

We are going to prove this theorem by means of some de la Vallée Poussin means of f, which are defined as follows

$$\mathcal{V}_n f(x) = \frac{1}{n} \sum_{k=n}^{2n-1} S_k f(x),$$

where $S_k f$ denotes the k-th Fourier–Jacobi partial sum of f associated with a suitable weight $w = v^{\gamma,\delta}$. In the sequel we briefly recall the main properties of $\mathcal{V}_n f$ we are going to use. For more details on de la Vallée Poussin type approximation, we refer the reader to [2, 3, 11, 21, 22, 23, 26, 27, 28, 29, 30].

Obviously, $\mathcal{V}_n f \in \mathbb{P}_{2n-1}$ for all f, and the following invariance on polynomials holds

$$\mathcal{V}_n P = P, \qquad \forall P \in \mathbb{P}_n. \tag{18}$$

Moreover, denoted by $K_n(w,x,y) := \sum_{j=0}^{n} p_j(w,x)p_j(w,y)$ the Darboux kernel of degree n corresponding to w, de la Vallée Poussin means of f associated with w can be explicitly written as follows

$$\mathcal{V}_n f(x) = \int_{-1}^{1} v_n(w,x,y)f(y)w(y)dy, \tag{19}$$

where $v_n(w,x,y) := \frac{1}{n}\sum_{r=n}^{2n-1} K_r(w,x,y)$.
Finally, by using a pointwise estimate of the previous de la Vallée Poussin kernel (see e.g. [30, Lemma 5.1]), the next theorem has been proved in [23, Th. 4.2].

Theorem 3. *Let $w = v^{\gamma,\delta}$ and $u = v^{\alpha,\beta}$ be such that $-1 < \alpha < \gamma$, $-1 < \beta < \delta$, and suppose they satisfy for some $\nu \in [0, 1/2]$ the following conditions*

$$\begin{aligned} \frac{\gamma}{2} - \frac{3}{4} - \nu \quad &< \alpha < \quad \frac{\gamma}{2} + \frac{1}{4} - \nu, \\ \frac{\delta}{2} - \frac{3}{4} - \nu \quad &< \beta < \quad \frac{\delta}{2} + \frac{1}{4} - \nu. \end{aligned} \tag{20}$$

Then for all $n \in \mathbb{N}$ and any $f \in L^1_u$, de la Vallée Poussin means associated with the weight w satisfy

$$\|(\mathcal{V}_n f - f)u\|_1 \le \mathcal{C} E_n(f)_u, \qquad \mathcal{C} \ne \mathcal{C}(n,f). \tag{21}$$

By taking into account the previous results, Theorem 2 can be proved as follows.

Proof of Theorem 2. Supposed that $u = v^{\alpha,\beta} \in L^1$ is arbitrarily fixed, let us take a weight $w = v^{\gamma,\delta}$ satisfying the hypotheses of Theorem 3, and let $\mathcal{V}_n f$ be the associated de la Vallée Poussin mean satisfying (21). By (1) we get

$$R_n(f)_u = R_n(f - \mathcal{V}_n f)_u \le \|(f - \mathcal{V}_n f)u\|_1 + \sum_{k=1}^{n} \lambda_k |f(x_k) - \mathcal{V}_n f(x_k)|. \tag{22}$$

On the other hand, set

$$\Gamma_t(x) = \begin{cases} 1 & x > t, \\ 0 & x \le t, \end{cases} \qquad\qquad x, t \in [-1, 1],$$

we observe that

$$f(x) = f(-1) + \int_{-1}^{1} \Gamma_t(x) df(t), \qquad x \in [-1, 1], \tag{23}$$

and, by applying (18) with $P(x) = f(-1)$, the identities (23) and (19) imply that

$$\begin{aligned} f(x) - \mathcal{V}_n f(x) &= [f(x) - f(-1)] - \mathcal{V}_n[f - f(-1)](x) \\ &= \int_{-1}^{1} \Gamma_t(x) df(t) - \int_{-1}^{1} \mathcal{V}_n \Gamma_t(x) df(t) \end{aligned}$$

holds for all $x \in [-1,1]$. Consequently, from (22) we deduce

$$R_n(f)_u \le \int_{-1}^{1} \|(\Gamma_t - \mathcal{V}_n\Gamma_t)u\|_1 |df(t)| + \int_{-1}^{1} \sum_{k=1}^{n} \lambda_k |\Gamma_t(x_k) - \mathcal{V}_n\Gamma_t(x_k)||df(t)|, \tag{24}$$

and in order to get the statement, in the sequel we are going to estimate the integrands in (24) for all $t \in [-1,1]$.

Let us first assume that $t \in [x_1, x_n]$. In this case we consider the so–called Markov–Stieltjes polynomials, $P_t^{\pm} \in \mathbb{P}_n$, which satisfy (see [13, p.72] and [24, Theorem 6.3.28])

$$P_t^-(x) \le \Gamma_t(x) \le P_t^+(x), \qquad \forall x, t \in [-1,1], \tag{25}$$

$$\|(P_t^+ - P_t^-)u\|_1 = \lambda_n(u,t) \le \frac{\mathcal{C}}{n}\, u(t)\varphi(t), \qquad \forall |t| \le 1 - \frac{\mathcal{C}}{n^2}, \tag{26}$$

where $\lambda_n(u,t) := \left[\sum_{j=0}^{n-1} p_j(u,t)^2\right]^{-1}$ and $\mathcal{C} \ne \mathcal{C}(n,t)$.

By (21), (25) and (26), we have

$$\begin{aligned} \|(\Gamma_t - \mathcal{V}_n\Gamma_t)u\|_1 &\le \mathcal{C}E_n(\Gamma_t)_u \le \mathcal{C}\|(\Gamma_t - P_t^-)u\|_1 \le \mathcal{C}\|(P_t^+ - P_t^-)u\|_1 \\ &\le \frac{\mathcal{C}}{n}\, u(t)\varphi(t). \end{aligned}$$

Similarly, by using (18), (25), (14), (21) and (26), we get

$$\begin{aligned} \sum_{k=1}^{n} \lambda_k |\Gamma_t(x_k) - \mathcal{V}_n\Gamma_t(x_k)| &\le \sum_{k=1}^{n} \lambda_k |\Gamma_t(x_k) - P_t^-(x_k)| + \sum_{k=1}^{n} \lambda_k |\mathcal{V}_n(\Gamma_t - P_t^-)(x_k)| \\ &\le \mathcal{C}\sum_{k=1}^{n} \lambda_k [P_t^+(x_k) - P_t^-(x_k)] + \mathcal{C}\|\mathcal{V}_n(\Gamma_t - P_t^-)u\|_1 \\ &\le \mathcal{C}\|(P_t^+ - P_t^-)u\|_1 + \mathcal{C}\|(\Gamma_t - P_t^-)u\|_1 \\ &\le \mathcal{C}\|(P_t^+ - P_t^-)u\|_1 \le \frac{\mathcal{C}}{n}\, u(t)\varphi(t). \end{aligned}$$

Now, let us assume $t \in (x_n, 1]$. Note that, by (21) we have

$$\|(\Gamma_t - \mathcal{V}_n\Gamma_t)u\|_1 \le \mathcal{C}E_n(\Gamma_t)_u \le \mathcal{C}\|\Gamma_t u\|_1 = \mathcal{C}\int_t^1 u(x)dx,$$

and taking into account that $1 \le (1+t) \le 2$ and $\sqrt{1-t} \le \sqrt{1-x_n} \le \mathcal{C}n^{-1}$, by easy computations we get

$$\|(\Gamma_t - \mathcal{V}_n\Gamma_t)u\|_1 \le \mathcal{C}\int_t^1 u(x)dx \le \frac{\mathcal{C}}{n}\, u(t)\varphi(t).$$

Similarly, by (14) and (21) we have

$$\begin{aligned}\sum_{k=1}^{n} \lambda_k |\Gamma_t(x_k) - \mathcal{V}_n\Gamma_t(x_k)| &= \sum_{k=1}^{n} \lambda_k |\mathcal{V}_n\Gamma_t(x_k)| \le \mathcal{C}\|(\mathcal{V}_n\Gamma_t)u\|_1 \le \mathcal{C}\|\Gamma_t u\|_1 \\ &\le \frac{\mathcal{C}}{n}\, u(t)\varphi(t).\end{aligned}$$

Finally, consider the symmetric case $t \in [-1, x_1)$. By using (21), and taking into account that $1 \le (1-t) \le 2$ and $\sqrt{1+t} \le \sqrt{1+x_1} \le \mathcal{C}n^{-1}$, we deduce

$$\|(\Gamma_t - \mathcal{V}_n\Gamma_t)u\|_1 \le \mathcal{C}E_n(\Gamma_t)_u \le \mathcal{C}\|(\Gamma_t - 1)u\|_1 = \mathcal{C}\int_{-1}^{t} u(x)dx \le \frac{\mathcal{C}}{n}\, u(t)\varphi(t),$$

as well as, by (18), (14) and (21) we get

$$\begin{aligned}\sum_{k=1}^{n} \lambda_k |\Gamma_t(x_k) - \mathcal{V}_n\Gamma_t(x_k)| &= \sum_{k=1}^{n} \lambda_k |\mathcal{V}_n(1-\Gamma_t)(x_k)| \le \mathcal{C}\|\mathcal{V}_n(1-\Gamma_t)u\|_1 \\ &\le \mathcal{C}\|(1-\Gamma_t)u\|_1 \le \frac{\mathcal{C}}{n}\, u(t)\varphi(t).\end{aligned}$$

Summing up, for all $t \in [-1, 1]$ we proved that there exists a constant $\mathcal{C} \ne \mathcal{C}(n, t)$ such that

$$\|(\Gamma_t - \mathcal{V}_n\Gamma_t)u\|_1 \le \frac{\mathcal{C}}{n}\, u(t)\varphi(t) \quad \text{and} \quad \sum_{k=1}^{n} \lambda_k |\Gamma_t(x_k) - \mathcal{V}_n\Gamma_t(x_k)| \le \frac{\mathcal{C}}{n}\, u(t)\varphi(t)$$

hold, hence by (24) the statement follows. □

4. Conclusions

We stated a new error bound (cf. (4)) for Gauss–Jacobi quadrature rules applied to functions in the weighted Besov spaces B_u^s given in (8) with $s \ge 1$.

Our estimate implies the already known error bounds (16) in Sobolev spaces, but in the intermediate Besov spaces B_u^s with $s > 1$, it has the advantage of assuring the same convergence rate of the error of best polynomial approximation in L_u^1 (Example 1, Corollary 1).

The case that $f \in B_u^s$ with $0 < s < 1$ remains an open problem. For the time being, an estimate is given for the functions of bounded variation (Theorem 2).

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